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On the distribution of zeros of solutions of a first order neutral differential equation

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Abstract

This paper is devoted to study the distribution of zeros of all solutions of the first-order neutral differential equation

$$[x(t) - px(t - \tau)]' + Q(t)x(t - \sigma) = 0, \; t > t_0,$$

where $p > 1$, $\tau, \sigma > 0$, and $Q \in C([t_0, \infty), (0, \infty))$.

We obtain new estimates for the distance between adjacent zeros of all solutions of the above equation under suitable criteria. Our results are supported with illustrative examples.

Keywords: Distribution of zeros; Oscillation, Neutral differential equations

Introduction

In this article, we estimate the distance between adjacent zeros of all solutions of the neutral equation

$$[x(t) - px(t - \tau)]' + Q(t)x(t - \sigma) = 0, \qquad (1)$$

where $\tau, \sigma > 0$, $p > 1$ and $Q \in C([t_0, \infty), (0, \infty))$. We shall assume that

$$\int_{t-\tau}^{t} Q_*(s)ds \geq \zeta_0, \quad \text{for } t \geq t_0 + 2\tau, \qquad (2)$$

for some $\zeta_0 > 0$ where $Q_*(t) = \min_{t-\tau \leq s \leq t}\{Q(s)\}$ for $t \geq t_0 + \tau$. By a solution of Eq.(1) we mean a function $x \in C([t_0 - \lambda, \infty), R)$, where $\lambda = \max\{\sigma, \tau\}$, such that $x(t) - px(t - \tau)$ is continuously differentiable and (1) is satisfied on $[t_0, \infty)$. We associate with (1) the initial condition $x(t) = \phi(t)$ on $[t_0 - \lambda, t_0]$ where $\phi \in C([t_0 - \lambda, t_0], R)$. The method of steps can be used to show that the resulting initial value problem has a unique solution. A solution x is said to be oscillatory if it has arbitrary large zeros. Equation (1) is called oscillatory if all its solutions are oscillatory.

Neutral differential equations appear in many applications from engineering, physics, economy and mathematical biology see (Gopalsamy, 1992; Hale and Lunel, 1993; Kolmanovskii and Myshkis, 1999; Kolmanovskii and Nosov, 1986). The oscillation theory of neutral differential equations has received a great deal of attention in recent years; see (Agarwal *et al.*, 2012; Bainov and Mishev, 1991; Erbe *et al.*, 1995; Györi and Ladas, 1991) for an account of this theory. However, little is known about estimating the distance between consecutive zeros of the solutions of these equations; see (Wenrui *et al.*, 2007; Wu *et al.*, 2007; Wu and Xu, 2004; Yong and Zhicheng, 1997; Zhou,1999) for some results of this type. To the best of our knowledge, (Wernui *et al.*, 2007) is the only published work on this topic for equation (1). Therefore, our main goal of this work is to obtain new estimates of the distance between adjacent zeros of all solutions of Eq.(1) under suitable criteria. We extended new techniques developed by (El-Morshedy, 2011) for the delay equation

$$x'(t) + P(t)x(t - \tau) = 0,$$

to the neutral equation (1) using new ideas from (Wernui *et al.*, 2007). Our work is concluded with some illustrative examples. Throughout this work; $d_s(x)$ denotes the least upper bound of the distances between adjacent zeros of any solution $x(t)$ of Eq.(1) on $[s, \infty)$.

II. PRELIMINARIES

Let $d = \zeta_0(\frac{1}{p+1})^{\frac{\sigma+\tau}{\tau}}$ and the sequences $\{a_n\}$ and $\{b_m\}$ be defined by

$$a_1 = \frac{p}{(p+1)^2}, \quad a_{n+1} = \frac{\frac{p}{(p+1)^2}}{1 - a_n(1+d)}, n = 1, 2, \dots$$

and

$$b_1 = 1 + d, \quad b_{m+1} = \frac{1+d}{1 - \frac{p}{(p+1)^2} b_m}, \quad m = 1, 2, \dots$$

These sequences are due to (Wernui *et al.*, 2007). They showed that the number $\Delta = 1 - \frac{4(1+d)p}{(p+1)^2}$ governs some basic properties of $\{a_n\}$ and $\{b_m\}$ as in the following result which is derived from ([Lemma 1] Wernui *et al.*, 2007) and [Rrmark 1].

Lemma 1 *The sequences $\{a_n\}$ and $\{b_m\}$ converge if and only if $\Delta \geq 0$. Moreover; if $\Delta \geq 0$ then $\lim_{n\to\infty} a_n = \frac{1-\sqrt{\Delta}}{2(1+d)}$, $\lim_{n\to\infty} b_m = \frac{1-\sqrt{\Delta}}{2p}(1+p)^2$ and there exists a positive integer N_1 such that*

$$\frac{1}{b_m} < \frac{p}{p+1} \quad \text{for } p > 1,\, m \geq N_1. \tag{3}$$

Next, for any solution $x(t)$ of Eq.(1), we define another related functions $y(t)$ and $z(t)$ as follows

$$x(t) = y(t)e^{\beta t} \quad \text{where } \beta = \frac{1}{\tau}\ln(p+1)(\text{ or } e^{\beta\tau} = p+1), \tag{4}$$

and

$$z(t) = \int_{t-\tau}^{t} y(s)ds. \tag{5}$$

Lemma 2 *([lemma 2] Wernui et al., 2007) Let $\Delta \geq 0$ and $x(t)$ be a solution of Eq.(1) on $[t_0, \infty)$. For some $T_1 \geq t_0 + 2\tau$, if there exist a positive integer $N \geq 2$ and $T_2 \geq T_1 + N\tau + \sigma + \lambda$ such that $x(t) > 0$ on $[T_1, T_2]$, then*

$$\frac{z(t)}{z(t-\tau)} > a_n, \quad \text{for } t \in [T_1 + \tau + \sigma + \lambda,\, T_2 - n\tau],$$

for some $n \leq N - 1$.

Lemma 3 *([lemma 3] Wernui et al., 2007) Let $\Delta \geq 0$ and $x(t)$ be a solution of Eq.(1) on $[t_0, \infty)$. For some $T_1 \geq t_0 + 2\tau$, if there exist a positive integer $N \geq 2$ and $T_2 \geq T_1 + N\tau + \sigma + \lambda$ such that $x(t) > 0$ on $[T_1, T_2]$, then*

$$\frac{z(t)}{z(t-\tau)} < \frac{1}{b_m}, \quad \text{for } t \in [T_1 + (m+1)\tau + \sigma + \lambda,\, T_2], \tag{6}$$

for some $m \leq N - 1$.

Consider a sequence $\{\tilde{b}_m\}$ defined as follows

$$\tilde{b}_1 = 1, \quad \tilde{b}_{m+1} = \frac{1+d}{1 - \frac{p}{(p+1)^2}\tilde{b}_m}, \quad m = 1, 2, \dots \tag{7}$$

This sequence differs with $\{b_m\}$ in the initial term and both have the same properties explained in Lemma 1 when $\Delta \geq 0$. That is $\lim_{n\to\infty} \tilde{b}_m = \frac{1-\sqrt{\Delta}}{2p}(1+p)^2$ and there exists a positive integer N_2 such that

$$\frac{1}{\tilde{b}_m} < \frac{p}{p+1} \quad \text{for } p > 1,\, m \geq N_2.$$

The use of $\{\tilde{b}_m\}$ enables us to prove the following version of Lemma 3.

Lemma 4 *Let $\Delta \geq 0$ and $x(t)$ be a solution of Eq.(1) on $[t_0, \infty)$. If there exist a positive integer $N \geq 2$, $T_1 \geq t_0 + 2\tau$ and $T_2 \geq T_1 + (N-1)\tau + 2\lambda$ such that $x(t) > 0$ on $[T_1, T_2]$, then*

$$\frac{z(t)}{z(t-\tau)} < \frac{1}{\tilde{b}_m}, \quad \text{for } t \in [T_1 + m\tau + 2\lambda,\, T_2], \tag{8}$$

for some $m \leq N - 1$.

Proof. If $x(t) > 0$ on $[T_1, T_2]$ for $T_1 \geq t_0 + 2\tau$, it follows from (4) that $y(t) > 0$, for $t \in [T_1, T_2]$ and hence (5) yields $z(t) > 0$ on $[T_1 + \tau, T_2]$. Integrating (1) from $t - \tau$ to t, we get

$$x(t) - (p+1)x(t-\tau) + px(t-2\tau)$$

$$+ \int_{t-\tau}^{t} Q(s)x(s-\sigma)ds = 0.$$

Using (4), we have

$$y(t) - y(t-\tau) + \frac{p}{(p+1)^2}y(t-2\tau)$$

$$+ e^{-\beta t}\int_{t-\tau}^{t} Q(s)y(s-\sigma)e^{\beta(s-\sigma)}ds = 0. \tag{9}$$

Since $z'(t) = y(t) - y(t-\tau)$, then (9) yields

$$z'(t) = -\frac{p}{(p+1)^2}y(t-2\tau)$$

$$-e^{-\beta t}\int_{t-\tau}^{t} Q(s)y(s-\sigma)e^{\beta(s-\sigma)}ds,$$

which implies

$$z'(t) < 0, \text{ for } t \in [T_1 + \tau + \lambda, T_2].$$

Integrating (9) from $t-\tau$ to t,

$$z(t)-z(t-\tau)+\frac{p}{(p+1)^2}z(t-2\tau)$$

$$+\int_{t-\tau}^{t}e^{-\beta s}ds\int_{s-\tau}^{s}Q(u)y(u-\sigma)e^{\beta(u-\sigma)}du=0.$$

That is,

$$z(t)-z(t-\tau)+\frac{p}{(p+1)^2}z(t-2\tau)$$

$$+\int_{t-\tau}^{t}Q_*(s)e^{-\beta(\tau+\sigma)}z(s-\sigma)ds\le 0, \qquad (10)$$

for $t\in[T_1+2\tau+\sigma,\, T_2]$. Hence

$$\frac{z(t)}{z(t-\tau)}<1=\frac{1}{\tilde{\mathrm{b}}_1},\quad \text{for } t\in[T_1+2\tau+\lambda,\, T_2].$$

This inequality implies that

$$z(t-2\tau)>\tilde{\mathrm{b}}_1 z(t-\tau), \qquad (11)$$

for $t\in[T_1+3\tau+\lambda,\, T_2]$. Also (10) yields

$$(1+d)z(t)-z(t-\tau)+\frac{p}{(p+1)^2}z(t-2\tau)<0,$$

for $t\in[T_1+2\tau+\sigma+\lambda,\, T_2]$. Now, combining this inequality with (11), it follows that

$$(1+d)z(t)-\left(1-\frac{p}{(p+1)^2}\tilde{\mathrm{b}}_1\right)z(t-\tau)<0,$$

or

$$(1+d)z(t)<\left(1-\frac{p}{(p+1)^2}\tilde{\mathrm{b}}_1\right)z(t-\tau),$$

for $t\in[T_1+2\tau+2\lambda,\, T_2]$. Rearranging,

$$\frac{z(t)}{z(t-\tau)}<\frac{1-\frac{p}{(p+1)^2}\tilde{\mathrm{b}}_1}{1+d}=\frac{1}{\tilde{\mathrm{b}}_2},\ t\in[T_1+2\tau+2\lambda,\, T_2].$$

Similarly, for $t\in[T_1+3\tau+2\lambda,\, T_2]$, one can see that

$$\frac{z(t)}{z(t-\tau)}<\frac{1-\frac{p}{(p+1)^2}\tilde{\mathrm{b}}_2}{(1+d)}=\frac{1}{\tilde{\mathrm{b}}_3}.$$

Repeating this argument till $m\le L-1$, we obtain

$$\frac{z(t)}{z(t-\tau)}<\frac{1}{\tilde{\mathrm{b}}_m},\quad \text{for } t\in[T_1+m\tau+2\lambda,\, T_2].$$

The proof is complete.

In the sequel, we consider the first order delay differential inequality

$$x'(t)-P(t)x(t+r)\ge 0, \qquad (12)$$

where $P\in([t_0,\,\infty),[0,\,\infty))$, $r>0$ and

$$\int_t^{t+r}P(s)ds\ge\rho,\quad t\ge t_0. \qquad (13)$$

for some constant $\rho>0$. We find some interesting results about the positivity of certain solution x of (12) on some bounded intervals. For an easy reference, a sequence $\{A_n(t)\}$ is defined as follows

$$\begin{aligned}A_0(t)&=P(t),\quad t\ge t_0\\ A_n(t)&=A_{n-1}(t)\int_t^{t+r}A_{n-1}(s)e^{\int_t^{s+r}A_{n-1}(u)du}ds,\\ &\quad\text{for}\quad t\ge t_0\ \text{ and } n=1,\,2,\ldots\end{aligned}$$

Lemma 5 *Let n be a positive integer such that*

$$\int_t^{t+r}A_n(s)ds\ge 1,\quad \textit{for all } t\ge t_0.$$

If $x(t)$ is a nondecreasing function on $[T_1,\,T_2+\delta]$ which satisfies (12) on $[T_1,\,T_2]$, then $x(t)$ cannot be positive on $[T_1,\,T_2]$, where $T_2>T_1+(3n+1)r-\delta$, $T_1\ge t_0$ and $|\delta|\le r$.

Proof. For the sake of contradiction, suppose that $x(t)$ is positive on $[T_1,\,T_2]$. Integrating (12) from t to $t+r$, we obtain

$$x(t+r)-x(t)-\int_t^{t+r}P(s)x(s+r)ds\ge 0,$$

for $t\in[T_1,\,T_2-r]$. Using this inequality and (12), we obtain

$$x'(t)-P(t)x(t)-P(t)\int_t^{t+r}P(s)x(s+r)ds\ge 0,$$

for $t\in[T_1,\,T_2-r]$. Let $y_1(t):=e^{-\int_{t_0}^{t}P(s)ds}x(t)$. Then $y_1(t)>0$ for $t\in[T_1,\,T_2]$ and the above inequality yields

$$y_1'(t)-P(t)\int_t^{t+r}P(s)e^{\int_t^{s+r}P(s)ds}y_1(s+r)ds\ge 0, \quad (14)$$

for $t\in[T_1,\,T_2-r]$. Since $x(t)$ is nondecreasing on $[T_1,\,T_2+\delta]$, it follows that

$$\begin{aligned}y_1'(t)&=(x'(t)-P(t)x(t))e^{-\int_{t_0}^{t}P(s)ds}\\&\ge(x'(t)-P(t)x(t+r))e^{-\int_{t_0}^{t}P(s)ds}\\&\ge 0,\quad \text{for } t\in[T_1,\,T_2+\delta-r].\end{aligned}$$

Thus (14) leads to

$$y_1'(t)-A_1(t)y_1(t+r)\ge 0,\quad \text{for } t\in[T_1,\,T_2+\delta-3r],$$

which has the same form of (12) but with different coefficient. So using similar arguments as those applied for (12), we obtain

$$y_1'(t) - A_1(t)y_1(t) - A_1(t)\int_t^{t+r} A_1(s)y_1(s+r)ds \geq 0,$$

for $t \in [T_1,\, T_2+\delta-4r]$. Set $y_2(t) := e^{-\int_{t_0}^t A_1(s)ds}y_1(t)$. Then

$$y_2'(t) - A_1(t)\int_t^{t+r} A_1(s)e^{\int_t^{s+r} A_1(s)ds}y_2(s+r)ds \geq 0,$$

for $t \in [T_1,\, T_2+\delta-4r]$, and thus $y_2'(t) \geq 0$ for $t \in [T_1,\, T_2+\delta-4r]$. Hence

$$y_2'(t) - A_2(t)y_2(t+r) \geq 0, \quad \text{for } t \in [T_1,\, T_2+\delta-6r].$$

So, an induction yields

$$y_n'(t) - A_n(t)y_n(t+r) \geq 0, \quad \text{for } t \in [T_1,\, T_2+\delta-3nr], \tag{15}$$

where $y_n'(t) \geq 0$ for $t \in [T_1,\, T_2+\delta-(3n-2)r]$. Integrate (15) from t to $t+r$, we obtain

$$y_n(t+r) - y_n(t) - \int_t^{t+r} A_n(s)y_n(s+r)ds \geq 0,$$

for $t \in [T_1,\, T_2+\delta-(3n+1)r]$.Therefore

$$0 < y_n(t) \leq \left[1 - \int_t^{t+r} A_n(s)ds\right] y_n(t+r) \leq 0,$$

for $t \in [T_1,\, T_2+\delta-(3n+1)r]$, which is impossible. The proof is complete.

In the next lemma, we give an interesting result that determine lower bound for the ratio $\frac{x(t+\tau)}{x(t)}$ by making use of the sequence $\{f_n(\rho)\}$ defined by (Xianhua and Jianshe, 1999), for $0<\rho<1$, as follows

$$f_0(\rho) = 1, \quad f_1(\rho) = \frac{1}{1-\rho},$$

$$f_{n+2}(\rho) = \frac{f_n(\rho)}{f_n(\rho) + 1 - e^{\rho f_n(\rho)}},\ n = 0,\, 1, \ldots \tag{16}$$

where ρ is defined by (13). According to (Xianhua and Jianshe, 1999), the sequence $f_{n+2}(\rho)$ could be positive, negative as some $n \geq 0$ or its denominator $(f_n(\rho) + 1 - e^{\rho f_n(\rho)})$ is zero. In the last case we say that $f_{n+2}(\rho) = \infty$.

Lemma 6 *Assume that (13) holds for $0<\rho<1$ and there exist $T_1 \geq t_0$, $|\delta| \leq r$, $T \geq T_1 + (n+1)r - \delta$ and a function $x(t)$ satisfying inequality (12) on $[T_1,\, T]$ with $x'(t) \geq 0$ for $t \in [T_1,\, T+\delta]$. If $x(t)$ is positive on $[T_1,\, T]$, then*

$$\frac{x(t+r)}{x(t)} \geq f_n(\rho) > 0 \text{ for } t \in [T_1, T-(n+1)r+\delta], \tag{17}$$

for some integer $n \geq 0$, where $f_n(\rho)$ is defined by (16).

Proof. Since $x(t)$ is nondecreasing on $[T_1,\, T+\delta]$. It follows that

$$\frac{x(t+r)}{x(t)} \geq 1 = f_0(\rho) \quad \text{for } t \in [T_1,\, T-r+\delta].$$

Integrating inequality (12) from t to $t+r$, we have

$$x(t+r) - x(t) \geq \int_t^{t+r} P(s)x(s+r)ds \geq \rho x(t+r),$$

for $t \in [T_1,\, T-2r+\delta]$.That is,

$$\frac{x(t+r)}{x(t)} \geq \frac{1}{1-\rho} = f_1(\rho) > 0 \text{ for } t \in [T_1,\, T-2r+\delta].$$

Now, when $t \in [T_1,\, T-3r+\delta]$, integrate inequality (12) from t to $t+r$,

$$x(t+r) \geq x(t) + \int_t^{t+r} P(s)x(s+r)ds. \tag{18}$$

Dividing both sides of (12) by $x(t)$ and integrating from $t+r$ to $s+r$, we obtain

$$\begin{aligned}\frac{x(s+r)}{x(t+r)} &\geq exp\Big(\int_{t+r}^{s+r} P(u)\frac{x(u+r)}{x(u)}du\Big)\\ &\geq exp\Big(f_0(\rho)\int_{t+r}^{s+r} P(u)du\Big).\end{aligned}$$

Using this inequality and (18), we get

$$\begin{aligned}x(t+r) &\geq x(t) + x(t+r)\int_t^{t+r} P(s)\frac{x(s+r)}{x(t+r)}ds\\ &\geq x(t) + x(t+r)\int_t^{t+r} P(s)\\ &\quad exp\Big(f_0(\rho)\int_{t+r}^{s+r} P(u)du\Big)ds\\ &= x(t) + x(t+r)\int_t^{t+r} P(s)exp\\ &\quad \left[f_0(\rho)\Big(\int_s^{s+r} P(u)du - \int_s^{t+r} P(u)du\Big)\right]ds\\ &\geq x(t) + x(t+r)e^{\rho f_0(\rho)}\int_t^{t+r} P(s)exp\\ &\quad \Big(-f_0(\rho)\int_s^{t+r} P(u)du\Big)ds\\ &= x(t) + \frac{x(t+r)e^{\rho f_0(\rho)}}{f_0(\rho)}\\ &\quad - \frac{x(t+r)e^{\rho f_0(\rho)}exp\Big(-f_0(\rho)\int_t^{t+r} P(u)du}{f_0(\rho)}\\ &= x(t) + \frac{x(t+r)(e^{\rho f_0(\rho)}-1)}{f_0(\rho)}.\end{aligned}$$

So

$$\frac{x(t+r)}{x(t)} \geq \frac{f_0(\rho)}{f_0(\rho)+1-e^{\rho f_0(\rho)}} = f_2(\rho) > 0,$$

for $t \in [T_1,\, T-3r+\delta]$. Repeating the above steps yields

$$\frac{x(t+r)}{x(t)} \geq f_n(\rho) > 0 \quad \text{for } t \in [T_1,\, T-(n+1)r+\delta].$$

The proof of Lemma 6 is complete.

Next, we need a sequence $\{q_n(s)\}_{n\geq 1}$ defined for $s \in (t,\, t+\tau)$ as follows

$$q_1(s) := P(s)$$
$$q_{n+1}(s) := P(s+nr)\int_s^{t+r} P_n(u)du,\ t \geq t_0,$$

and we consider that $\sum_{k=a}^{b} L_k = 1$ for any sequence $\{L_n\}$ as long as $b < a$.

Lemma 7 *Let n^* and n^{**} be two positive integers such that $n^{**} = \min\{l : f_{l+1}(\rho) < 0$ or $f_{l+1}(\rho) = \infty\}$ and*

$$\sum_{k=1}^{n^*}\left(\prod_{i=2}^{k} f_{n^*+2-i}(\rho)\right)\int_t^{t+r} q_k(s)ds \geq 1, \quad t \geq t_0,$$

for $\rho \in (0,\, 1)$. Further; assume that $x(t)$ is nondecreasing on $[T_1,\, T_2+\delta]$, where $T_1 \geq t_0$ and $|\delta| \leq r$. If $x(t)$ satisfies (12) on $[T_1,\, T_2]$, then $x(t)$ cannot be positive on $[T_1,\, T_2]$, where $T_2 > T_1 + (n+2)r - \delta$ and $n = \min\{n^*,\, n^{**}\}$.

Proof. Suppose, for the sake of contradiction, that $x(t)$ is positive on $[T_1,\, T_2]$. If $n = n^{**}$, then Lemma 6 implies a contradiction. Thus, assume that $n = n^*$ and integrate (12) from t to $t+r$, we have

$$x(t+r) - x(t) - \int_t^{t+r} P(s)x(s+r)ds \geq 0, \tag{19}$$

for $t \in [T_1,\, T_2 - r]$. Using integration by parts, we obtain

$$\begin{aligned}&\int_t^{t+r} P(s)x(s+r)ds\\ &= \int_t^{t+r} x(s+r)d(-\int_s^{t+r} P(u)du) \geq x(t+r)\\ &\int_t^{t+r} q_1(s)ds + \int_t^{t+r} q_2(s)x(s+2r)\,\mathrm{ds}\\ &\geq \Big(\int_t^{t+r} q_1(s)ds\Big)x(t+r) + \Big(\int_t^{t+r} q_2(s)ds\Big)x(t+2r)\\ &\int_t^{t+r} q_3(s)x(s+3r)ds, \quad \text{for } t \in [T_1,\, T_2-3r].\end{aligned}$$

Continuing the above arguments n times, we find

$$\int_t^{t+r} P(s)x(s+r)ds \geq \sum_{k=1}^{n}\Big(\int_t^{t+r} q_k(s)ds\Big)x(t+kr)$$
$$+\int_t^{t+r} q_{n+1}(s)x(s+(n+1)r)ds,\ t \in [T_1,\, T_2-(n+1)r].$$

Since $x(s+(n+1)r) > 0$ for $s \in [T_1,\, T_2+\delta-(n+1)r]$, then

$$\int_t^{t+r} P(s)x(s+r)ds \geq \sum_{k=1}^{n}\Big(\int_t^{t+r} q_k(s)ds\Big)x(t+kr), \tag{20}$$

for $t \in [T_1,\, T_2+\delta-(n+2)r]$. On the other hand, for $t \in [T_1,\, T_2+\delta-(n+2)r]$, we have $x(t) > 0$ and

$$t+(i-1)r \in [T_1,\, T_2+\delta-(n+3-i)r], \quad i = 2,\, 3, \ldots,\, n.$$

So, when $t \in [T_1,\, T_2+\delta-(n+2)r]$, Lemma 6 implies that

$$\frac{x(t+ir)}{x(t+(i-1)r)} \geq f_{n+2-i}(\rho), \quad i = 2,\, 3, \ldots,\, n.$$

Therefore,

$$\begin{aligned}x(t+kr) &= \Big(\prod_{i=2}^{k}\frac{x(t+ir)}{x(t+(i-1)r)}\Big)x(t+r)\\ &\geq \Big(\prod_{i=2}^{k} f_{n+2-i}(\rho)\Big)x(t+r),\ k = 1,\, 2, \ldots,\, n.\end{aligned}$$

Using the above inequality and (20), we get

$$\begin{aligned}\int_t^{t+r} P(s) \quad x(s+r)ds &\geq \sum_{k=1}^{n}\Big(\int_t^{t+r} q_k(s)ds\Big)x(t+kr)\\ &\geq \sum_{k=1}^{n}\Big(\prod_{i=2}^{k} f_{n+2-i}(\rho)\int_t^{t+r} q_k(s)ds\Big)x(t+r),\end{aligned}$$

for $t \in [T_1,\, T_2+\delta-(n+2)r]$. Thus (19) leads to

$$x(t) \leq \left[1 - \sum_{k=1}^{n}\prod_{i=2}^{k} f_{n+2-i}(\rho)\int_t^{t+r} q_k(s)ds\right]x(t+r) \leq 0.$$

This contradiction completes the proof.

III. MAIN RESULTS

In the following results, we restrict ourselves to the cases when $\Delta \geq 0$ and $\tau > \sigma$. It will be assumed that

$$\int_t^{t+\tau-\sigma} Q_*(s)ds \geq \zeta_1, \quad t \geq t_1 \tag{21}$$

for some positive constant ζ_1 and $t_1 \geq t_0 + 2\tau$.

In view of Lemma 1 we have $\lim_{n\to\infty} a_n = \frac{1-\sqrt{\Delta}}{2(1+d)}$. So the sequence $\{\eta_n\}$ where $\eta_n = \frac{\zeta_1}{p-a_n(p+1)}$, $n > 0$, converges to a number $\eta = \frac{\zeta_1}{p-a^0(p+1)}$. We define a sequence

$\{A_{n,j}(t)\}$ for any positive integer j by

$$A_{0,j}(t) = \frac{Q_*(t)}{p - a_j(p+1)}, \quad t \geq t_1$$
$$A_{n,j}(t) = A_{n-1,j}(t) \int_t^{t+\tau-\sigma} A_{n-1,j}(s) exp\Big(\int_t^{s+\tau-\sigma} A_{n-1,j}(u) du\Big) ds,\ t \geq t_1,\ n = 1, 2, \ldots$$

Theorem 1 *Assume that (21) holds. Let n be a positive integer such that*

$$\int_t^{t+\tau-\sigma} A_{n,j^*}(s) ds \geq 1, \quad t \geq t_1. \qquad (22)$$

Then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m' + j^* + 3)\tau + 3n(\tau - \sigma)$, where

$$j^* = \min\{\ j | \eta_j > 0\ \}, \qquad (23)$$

and

$$m' = \min_{m \geq 1} \left\{ m | \frac{1}{b_m} < \frac{p}{p+1} \right\}. \qquad (24)$$

Proof. Let $x(t)$ be a solution of Eq.(1) with $x(t) > 0$ on $[T_1, T_2]$ where $T_2 > T_1 + (m'+2)\tau + j^*\tau + \sigma + (3n+1)(\tau - \sigma)$, $T_1 \geq t_1$. In view of (4), Eq.(1) implies that

$$\left[y(t) - \frac{p}{p+1} y(t-\tau) \right]' + \beta \left[y(t) - \frac{p}{p+1} y(t-\tau) \right]$$
$$+ Q(t) e^{-\beta\sigma} y(t-\sigma) = 0.$$

Integrating from $t - \tau$ to t, we get

$$\left[z(t) - \frac{p}{p+1} z(t-\tau) \right]' + \beta \left[z(t) - \frac{p}{p+1} z(t-\tau) \right]$$
$$+ e^{-\beta\sigma} \int_{t-\tau}^t Q(s) y(s-\sigma) ds = 0,$$

which yields

$$\left[z(t) - \frac{p}{p+1} z(t-\tau) \right]' + \beta \left[z(t) - \frac{p}{p+1} z(t-\tau) \right]$$
$$+ e^{-\beta\sigma} Q_*(t) z(t-\sigma) \leq 0 \quad \text{for } t \in [T_1 + \tau + \sigma, T_2]. \quad (25)$$

Since $\Delta \geq 0$ and $p > 1$, it follows from (3) and Lemma 3 that

$$\frac{z(t)}{z(t-\tau)} < \frac{1}{b_m} < \frac{1}{b_{m'}} < \frac{p}{p+1}, \qquad (26)$$

for any $m \geq m'$ where m' is defined by (24) and $t \in [T_1 + (m'+2)\tau + \sigma, T_2]$. Set

$$u(t) = z(t) - \frac{p}{p+1} z(t-\tau), \quad t \in [T_1 + (m'+2)\tau + \sigma, T_2]. \qquad (27)$$

Then (26) gives

$$u(t) < 0, \quad \text{for } t \in [T_1 + (m'+2)\tau + \sigma, T_2].$$

Moreover; Lemma 2 yields

$$\begin{aligned} u(t) &= z(t) - \frac{p}{p+1} z(t-\tau) \\ &= \left[\frac{z(t)}{z(t-\tau)} - \frac{p}{p+1} \right] z(t-\tau) \\ &> \left[a_j - \frac{p}{p+1} \right] z(t-\tau), \end{aligned}$$

for $t \in [T_1 + 2\tau + \sigma, T_2 - j\tau]$ and any positive integer j. Thus

$$u(t + \tau - \sigma) > \left[a_j - \frac{p}{p+1} \right] z(t-\sigma), \qquad (28)$$

for $t \in [T_1 + 2\tau + \sigma, T_2 - (j+1)\tau + \sigma]$. Substituting (27) and (28) into (25),

$$u'(t) + \beta u(t) - \frac{Q_*(t) e^{-\beta\sigma}}{\frac{p}{p+1} - a_j} u(t + \tau - \sigma) < 0, \qquad (29)$$

for $t \in [T_1 + 2\tau + \sigma, T_2 - (j+1)\tau + \sigma]$. Put $w(t) = -e^{\beta t} u(t)$, for $t \in [T_1 + (m'+2)\tau + \sigma, T_2]$ and take $j = j^*$, $T = T_1 + (m'+2)\tau + \sigma$, then

$$w(t) > 0, \quad \text{for } t \in [T, T + 3n(\tau - \sigma)], \qquad (30)$$

and inequality (25) leads to

$$\begin{aligned} w'(t) &= -\big(u'(t) + \beta u(t)\big) e^{\beta t} \\ &\geq e^{\beta(t-\sigma)} Q_*(t) z(t-\sigma) > 0,\ t \in [T - m'\tau, T_2]. \end{aligned}$$

Hence $w't) > 0$ on $[T, T + (3n+1)(\tau - \sigma)]$ and (29) is transformed to the form

$$w'(t) - \frac{Q_*(t)}{p - a_{j^*}(p+1)} w(t + \tau - \sigma) > 0,$$

for $t \in [T, T + 3n(\tau - \sigma)]$. Using Lemma 5, with $\delta = \tau - \sigma$ and A_n is replaced by $A_{n,j}$, we conclude that $w(t)$ cannot be positive on $[T, T + 3n(\tau - \sigma)]$. This contradiction completes the proof.

Corollary 1 *Assume that (21) holds. Let $\{\alpha_{n,j^*}\}_{n \geq 1}$ be a sequence defined by*

$$\alpha_{n,j^*} = \alpha_{n-1,j^*}(e^{2\alpha_{n-1,j^*}} - e^{\alpha_{n-1,j^*}}), \quad \alpha_{0,j^*} = \eta_{j^*}. \qquad (31)$$

If there exists a positive integer n_0 satisfying $\alpha_{n_0,j^*} \geq 1$, then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m' + 3 + j^*)\tau + 3n_0(\tau - \sigma)$, where j^* is defined by (23) and m' is defined by (24).

Proof. From (21), it follows that

$$\int_t^{t+\tau-\sigma} A_{0,j^*}(s)ds \geq \alpha_{0,j^*} = \eta_{j^*}.$$

Thus

$$\begin{aligned}
&\int_t^{t+\tau-\sigma} A_{1,j^*}(s)ds \\
&= \int_t^{t+\tau-\sigma} A_{0,j^*}(s) \int_s^{s+\tau-\sigma} A_{0,j^*}(u) \\
&\quad e^{\int_s^{u+\tau-\sigma} A_{0,j^*}(v)dv} duds \\
&= \int_t^{t+\tau-\sigma} A_{0,j^*}(s) \int_s^{s+\tau-\sigma} A_{0,j^*}(u) e^{\int_s^{u} A_{0,j^*}(v)dv} \\
&\quad e^{\int_u^{u+\tau-\sigma} A_{0,j^*}(v)dv} duds \\
&\geq e^{\alpha_{0,j^*}} \int_t^{t+\tau-\sigma} A_{0,j^*}(s) \\
&\quad \Big(\int_s^{s+\tau-\sigma} A_{0,j^*}(u) e^{\int_s^{u} A_{0,j^*}(v)dv} du \Big) ds \\
&= e^{\alpha_{0,j^*}} \int_t^{t+\tau-\sigma} A_{0,j^*}(s) \Big(e^{\int_s^{s+\tau-\sigma} A_{0,j^*}(v)dv} - 1 \Big) ds \\
&\geq e^{\alpha_{0,j^*}} (e^{\alpha_{0,j^*}} - 1) \int_t^{t+\tau-\sigma} A_{0,j^*}(s)ds \geq \alpha_{1,j^*}.
\end{aligned}$$

By induction, it follows that

$$\int_t^{t+\tau-\sigma} A_{n,j^*}(s)ds \geq \alpha_{n,j^*}, \quad \text{for } n = 0, 1, \ldots$$

Thus, when $\alpha_{n_0,j^*} \geq 1$, then (22) holds for $n = n_0$. Therefore, the proof can be completed by applying Theorem 1.

In the following result, we consider the sequence $\{q_{n,j}(s)\}$ defined by

$$q_{1,j}(s) = \frac{Q_*(s)}{p - a_j(p+1)},$$
$$q_{n+1,j}(s) = q_{1,j}(s + n(\tau-\sigma)) \int_s^{t+\tau-\sigma} q_{n,j}(u)du,$$

for $t \geq t_1$ and $n \geq 1$.

Theorem 2 *Assume that (21) holds. If* n^*, n^{**} *are two positive integers such that* $n^{**} = \min\{l : f_{l+1}(\eta_{j^*}) < 0 \, or \, f_{l+1}(\eta_{j^*}) = \infty\}$ *and*

$$\sum_{k=1}^{n^*} \left(\prod_{i=2}^{k} f_{n^*+2-i}(\eta_{j^*}) \right) \int_t^{t+\tau-\sigma} q_{k,j^*}(s)ds \geq 1, \quad (32)$$

for $t \geq t_1$ *and* $\eta_{j^*} \in (0, 1)$, *then Eq.(1) is oscillatory and* $d_{t_1}(x) \leq (m' + 3 + j^*)\tau + (n+1)(\tau - \sigma)$ *for any solution* $x(t)$ *of (1), where* j^*, m' *are defined by (23), (24) respectively and* $n = \min\{n^*, n^{**}\}$.

Proof. Assume, for the sake of contradiction, that $x(t) > 0$ for $t \in [T_1, T_1 + (m'+2)\tau + \sigma + j^*\tau + (n+2)(\tau-\sigma)]$. We take $n = n^*$ since otherwise a contradiction appears. Proceeding as in Theorem 1, we obtain the inequality

$$w'(t) - \frac{Q_*(t)}{p - a_{j^*}(p+1)} w(t + \tau - \sigma) > 0,$$

for $t \in [T, T + (n+1)(\tau - \sigma)]$, and

$$w't) > 0, \quad \text{for } t \in [T, T + (n+2)(\tau - \sigma)].$$

Moreover, (30) yields

$$w(t) > 0 \quad \text{for } t \in [T, T + (n+1)(\tau - \sigma)].$$

According to Lemma 7, with $\delta = \sigma - \tau$ and q_n is replaced by $q_{n,j}$, we obtain that $w(t)$ cannot be positive on $[T, T + (n+1)(\tau - \sigma)]$. This contradiction completes the proof.

Since $f_2(\eta_{j^*}) = \frac{1}{2 - e^{\eta_{j^*}}}$ for $\eta_{j^*} \in (0, 1)$, then Theorem 2 and definition of n^{**} lead to the particular result.

Corollary 2 *Assume that (21) holds. If either* $\ln 2 \leq \eta_{j^*} < 1$ *or* $0 < \eta_{j^*} < \ln 2$ *and*

$$\int_t^{t+\tau-\sigma} q_{1,j^*}(s)ds + f_2(\eta_{j^*}) \int_t^{t+\tau-\sigma} q_{1,j^*}(s + \tau - \sigma) \int_s^{t+\tau-\sigma} q_{1,j^*}(u)duds \geq 1, \quad t \geq t_1, \quad (33)$$

then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m' + 3 + j^*)\tau + 3(\tau - \sigma)$, *where* j^*, m' *are defined by (23), (24) respectively.*

Corollary 3 *Assume that (21) holds and*

$$\frac{\zeta_1}{p - a_j(p+1)} \geq \eta_{j^*} \quad \text{for } \eta_{j^*} \in (0, 1). \quad (34)$$

If n^*, n^{**} *are two positive integers such that* $n^{**} = \min\{l : f_{l+1}(\eta_{j^*}) < 0 \, or \, f_{l+1}(\eta_{j^*}) = \infty\}$ *and*

$$\sum_{k=1}^{n^*} \prod_{i=2}^{k} f_{n^*-(i-2)}(\eta_{j^*}) \frac{\eta_{j^*}^k}{k!} \geq 1, \quad (35)$$

then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m' + 3 + j^*)\tau + (n+1)(\tau - \sigma)$, *where* j^*, m' *are defined by (23), (24) respectively and* $n = \min\{n^*, n^{**}\}$.

Example 1 *Consider the neutral differential equation*

$$[x(t) - 5.6\,x(t-3)]' + \left(2.6 + e^{-5t}\right) x(t-2) = 0, \quad t > 0,$$

which has the form (1) with $p = 5.6$, $Q(t) = 2.6 + e^{-5t}$, $\tau = 3$ *and* $\sigma = 2$. For $t \geq t_1 = 6$, we have

$$\int_{t-\tau}^{t} Q_*(s)ds \geq 7.8 = \zeta_0,$$

$$\int_{t}^{t+\tau-\sigma} Q_*(s)ds \geq 2.6 = \zeta_1,$$

$$d = 0.33588..., \text{ and } \Delta > 0.$$

So, it follows

$$a_1 = 0.12855..., \quad b_1 = 1.33588..., \quad \eta_1 = 0.5471...,$$
$$f_1(\eta_1) = 2.2084..., \quad f_2(\eta_1) = 3.6818...$$

Clearly, $\frac{1}{b_1} < \frac{p}{p+1}$, $m' = 1$, $j^* = 1$ and $\eta_1 + f_2(\eta_1)\,\frac{\eta_1^2}{2} = 1.09$. Applying Corollary 3, it follows that (35) holds for $n = n^* = 2$ and $d_6(x) \leq (m' + j^* + 3)\tau + (n+1)(\tau - \sigma) = 18$. Theorem 3 in (Wenrui *et al.*, 2007) can not give an estimation smaller than 18.

All previous results of this section can be proved using $\{\tilde{b}_m\}$ instead of $\{b_m\}$. The obtained results lead to better estimates in some situations. Next, we list those new results without proofs.

Theorem 3 *Assume that (21) holds. Let n be a positive integer such that (22) holds. Then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m'' + j^* + 2)\tau + (3n+1)(\tau - \sigma)$, where j^* is defined by (23) and*

$$m'' = \min_{m \geq 2} \left\{ m | \frac{1}{\tilde{b}_m} < \frac{p}{p+1} \right\}. \tag{36}$$

Corollary 4 *Assume that (21) holds. Let $\{\alpha_{n,j^*}\}$ be defined by (31). If there exists a positive integer n_0 satisfying $\alpha_{n_0,j^*} \geq 1$, then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m'' + 2 + j^*)\tau + (3n_0 + 1)(\tau - \sigma)$, where j^* is defined by (23) and m'' is defined by (36).*

Theorem 4 *Assume that (21) holds. If n^*, n^{**} are two positive integers such that $n^{**} = \min\{l : f_{l+1}(\eta_{j^*}) < 0 \, or \, f_{l+1}(\eta_{j^*}) = \infty\}$ and (32) holds for $\eta_{j^*} \in (0, 1)$, then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m'' + 2 + j^*)\tau + (n + 2)(\tau - \sigma)$ for any solution $x(t)$ of (1), where j^*, m'' are defined by (23), (36) respectively and $n = \min\{n^*, n^{**}\}$.*

Corollary 5 *Assume that (21) holds. If either $\ln 2 \leq \eta_{j^*} < 1 \, or \, 0 < \eta_{j^*} < \ln 2$ and (33) holds, then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m''+2+j^*)\tau + 4(\tau - \sigma)$, where j^*, m'' are defined by (23), (36) respectively.*

Corollary 6 *Assume that (21) and (34) hold. If n^*, n^{**} are two positive integers such that $n^{**} = \min\{l : f_{l+1}(\eta_{j^*}) < 0 \, or f_{l+1}(\eta_{j^*}) = \infty\}$ and (35) holds, then Eq.(1) is oscillatory and $d_{t_1}(x) \leq (m'' + 2 + j^*)\tau + (n + 2)(\tau - \sigma)$, where j^*, m'' are defined by (23), (36) respectively and $n = \min\{n^*, n^{**}\}$.*

Example 2 *Consider the neutral differential equation*

$$[x(t) - 3\,x(t-2)]' + \left(1.2 + \frac{2}{t}\right) x(t-1) = 0, \quad t > 0,$$

Then $p = 3$, $Q(t) = 1.2 + \frac{2}{t}$, $\tau = 2$, $\sigma = 1$ and for $t \geq t_1 = 4$, we have

$$\int_{t-\tau}^{t} Q_*(s)ds \geq 2.4 = \zeta_0,$$

$$\int_{t}^{t+\tau-\sigma} Q_*(s)ds \geq 1.2 = \zeta_1,$$

$$d = 0.3..., \text{ and } \Delta > 0.$$

In addition, we note that

$$a_1 = 0.1875..., \quad b_1 = 1.3, \; b_2 = 1.719..., \quad \tilde{b}_1 = 1,$$
$$\tilde{b}_2 = 1.6, \; \eta_1 = 0.5333..., \quad f_1(\eta_1) = 2.1427...,$$
$$f_2(\eta_1) = 3.3846...$$

Therefore, $m' = m'' = 2$, $j^* = 1$. Applying Corollary 6, we find that $\eta_1 + f_2(\eta_1)\,\frac{\eta_1^2}{2} > 1$. Then $n = n^* = 2$ and $d_4(x) \leq (m'' + j^* + 2)\tau + (n+2)(\tau - \sigma) = 14$. One can see that Corollary 3 gives that $d_4(x) \leq (m' + j^* + 3)\tau + (n+1)(\tau - \sigma) = 15$.

Remark 1 *Two sets of results are obtained. One using $\{b_m\}$ and the other using the modified sequence $\{\tilde{b}_m\}$. In example 2, it was shown that Corollary 6 gives better estimates than Corollary 3. But if we apply Corollary 6 to example 1, it is easy to see that $d_6(x) \leq 19$. So Corollary 3 leads to better estimates in this case. This emphasizes the importance of our new sequence $\{\tilde{b}_m\}$ and shows that these sets are independent of each other.*

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الملخص العربى

توزيع الأصفار لحلول معادلة تفاضلية محايدة من الدرجة الأولى

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سوف نناقش مسألة تحديد المسافة بين الأصفار المتجاورة لجميع الحلول لمعادلة تفاضلية ذات معامل تأخير من الشكل

$$[x(t) - p\,x(t-\tau)]' + Q(t)x(t-\sigma) = 0, \quad t \geq t_0,$$

حيث

$$p > 1,\ \tau, \sigma > 0$$
$$Q \in C([t_0,\ \infty), (0,\ \infty)$$

قدمنا الشروط الكافية للتذبذب لهذا النوع من المعادلات. وأعطينا بعض الأمثلة لتوضيح نتائجنا مع مقارنتها بنتائج سابقة.